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Boundary Value Problems and Best Approximation by Holomorphic Functions

ELIAS WEGERT

Sektion Mathematik, Bergakademie Freiberg, Freiberg, DDR-9200, East Germany

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1. INTRODUCTION

A classical problem in complex analysis consists in finding the distance of a function $f \in L^\infty$ on the unit circle $\mathbb{T}$ to H^∞, the space of functions which extend to a bounded holomorphic function in the unit disk $\mathbb{D}$. It is closely related to some other questions, such as the Pick–Nevanlinna problem of minimizing the supremum norm over the set of bounded holomorphic functions in $\mathbb{D}$, subject to a finite or infinite set of interpolation conditions [8–10] or the problem of seeking the largest circular domain of a positive harmonic function whose first Taylor coefficients are given [2].

The problem has a remarkable variety of applications, especially in systems engineering. Recent heightening of theoretical interest was brought about by results of Adamyan, Arov, and Krejn [1] on an equivalent problem in operator theory. Nowadays a series of related interpolation and approximation problems can be handled by several alternative mathematical approaches in a unified treatment (problems with matrix-valued functions included, cf. [7, 14], introduction, for instance).

Much less is known about a far-reaching generalization of the above problem, which was brought into discussion in the recent paper [6] by J. W. Helton and R. E. Howe (Unfortunately, this paper is not available to me at present, therefore I refer to [5]). The authors study the following optimization problem: Given a function $F\colon \mathbb{T} \times \mathbb{C}^N \to \mathbb{R}$, find

$$\inf_{w \in E} \sup_{t \in \mathbb{T}} F(t, w(t)), \tag{1.1}$$

where $E = (H^\infty \cap C)^N$ denotes the space of all continuous $\mathbb{C}^N$-valued functions on $\mathbb{T}$ with holomorphic continuation into $\mathbb{D}$. Assuming the existence

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of a continuous optimum w_0 for (1.1) they show that in effect this solution can be characterized by two properties:

$$(1) \quad F(t, w_0(t)) = \text{const}, \tag{1.2}$$

$$(2) \quad \text{wind}\, \frac{\partial F}{\partial w}(\cdot, w_0(\cdot)) = k \geqslant 1 \tag{1.3}$$

($N = 1$, wind denotes the winding number about the origin of a complex-valued nonvanishing function on $\mathbb{T}$). Under the assumptions that $F(t, f(t)) = 0$ and $F(t, w) > 0$ for $w \neq f(t)$, it makes sense to consider (1.1) as a generalized distance of the function f to E, where $E = H^\infty$ is of particular interest.

In the present paper we propose a new approach for determining (1.1) (with $N = 1$ and $E = H^\infty$), which is based on recent results about parameter-depending boundary value problems [13]. This method gives not only the characterization (1.2), (1.3) but also the existence and uniqueness of a continuous optimum w_0 under some quite general assumptions on the distance function F. Note that w_0 is the optimum over the whole space H^∞ and not only over $H^\infty \cap C$. Moreover, we prove Helton's conjecture [5, p. 362] that $k = 1$ in (1.3) for generic functions F.

2. Boundary Value Problems of Riemann–Hilbert Type

We begin by sketching some ideas concerning a class of nonlinear boundary value problems of complex analysis. Let $\{M_t\}_{t \in \mathbb{T}}$ denote a family of curves in the complex plane. We introduce the set $\mathfrak{M}$ of all manifolds

$$M := \bigcup_{t \in \mathbb{T}} \{t\} \times M_t \subset \mathbb{T} \times \mathbb{C} \tag{2.1}$$

subject to the following hypotheses.

(i) For each $t \in \mathbb{T}$ the curve M_t is homeomorphic to $\mathbb{T}$.

(ii) The manifold M is a C^1-submanifold of $\mathbb{T} \times \mathbb{C}$.

(iii) M is transversal to each plane $\{t\} \times \mathbb{C}$ ($t \in \mathbb{T}$).

For given $M \in \mathfrak{M}$, the following boundary value problem is considered: Find all functions $w = u + iv \in H^\infty \cap C$ holomorphic in the unit disk $\mathbb{D}$ which satisfy the boundary relation

$$w(t) \in M_t, \qquad \forall t \in \mathbb{T}. \tag{2.2}$$

Problems of this type are frequently called Riemann–Hilbert problems (RHP). In this paper we only deal with problems pertaining to closed curves M_t. Another class of problems addresses open curves M_t.

The list of references concerning RHP's for closed curves M_t is comparably small (an exception is, of course, the problem of conformal mapping involved in this case). Above all a paper by A. I. Šnirel'man [11] must be mentioned. Šnirel'man describes the solution set under the additional assumption that $0 \in \text{int}\, M_t$, $\forall t \in \mathbb{T}$. Further generalizations are due to M. A. Efendiev [3, 4]. In [12], the author proved existence results by means of Leray–Schauder techniques and discussed the influence of the condition $0 \in \text{int}\, M_t$. Furthermore, in [13], a connection between RHP's and a class of extremal problems was pointed out.

Before summarizing relevant results, some notations must be introduced. We denote the bounded and the unbounded component of $\mathbb{C} \setminus M_t$ by $\text{int}\, M_t$ and $\text{ext}\, M_t$, respectively. A similar definition is made for $\text{int}\, M$ and $\text{ext}\, M$. Further, for each $\varepsilon \geqslant 0$, let

$$\text{int}_\varepsilon M_t := \text{int}\, M_t \cup \{w \in \mathbb{C} : \text{dist}(w, M_t) < \varepsilon\},$$

$$\text{int}_{-\varepsilon} M_t := \text{int}\, M_t \cap \{w \in \mathbb{C} : \text{dist}(w, M_t) > \varepsilon\}.$$

For $\varepsilon \in \mathbb{R}$ we put

$$\text{int}_\varepsilon M := \bigcup_{t \in \mathbb{T}} \{t\} \times \text{int}_\varepsilon M_t.$$

If $M_0 \in \mathfrak{M}$ and $\varepsilon > 0$ then

$$U_\varepsilon(M_0) := \{M \in \mathfrak{M} \colon \text{int}_{-\varepsilon} M_0 \subset \text{int}\, M \subset \text{int}_\varepsilon M_0\}.$$

The local base $\{U_\varepsilon(M_0)\}_{\varepsilon > 0}$ of neighborhoods of M_0 makes $\mathfrak{M}$ become a Hausdorff space.

We define the trace $\text{tr} f$ of a function $f \in C(\mathbb{T})$ by

$$\text{tr} f := \bigcup_{t \in \mathbb{T}} \{t\} \times \{f(t)\} \subset \mathbb{T} \times \mathbb{C}.$$

Note that every function $w \in H^\infty \cap C$ is uniquely determined in $\mathbb{D}$ by its trace through the Poisson integral formula. The boundary condition (2.2) can now be written in the form

$$\text{tr}\, w \subset M. \tag{2.3}$$

For any solution $w \in H^\infty \cap C$ of the RHP (2.3) we define the winding number

$$\text{wind}_M w := \text{wind}(w - m) \tag{2.4}$$

of w with respect to M. Here $m \in C(\mathbb{T})$ is an arbitrary continuous function with $\operatorname{tr} m \subset \operatorname{int} M$. The "wind" on the right of (2.4) refers to the usual winding number about the origin. The solution set $W(M) \subset H^\infty \cap C$ of (2.3) splits into the classes

$$W_k(M) := \{w \in W(M) : \operatorname{wind}_M w = k\}, \qquad k \in \mathbb{Z}.$$

If no confusion is possible, we sometimes write W and W_k instead of $W(M)$ and $W_k(M)$, respectively.

With regard to the solvability of the RHP, the following definition is given: The manifold $M \in \mathfrak{M}$ is called regularly (holomorphically) traceable if there exists a function $w_0 \in H^\infty \cap C$ with

$$\operatorname{tr} w_0 \subset \operatorname{int} M. \tag{2.5}$$

The manifold $M \in \mathfrak{M}$ is said to be singularly (holomorphically) traceable if it is not regularly traceable but there exists a $w_0 \in H^\infty \cap C$ with

$$\operatorname{tr} w_0 \subset \operatorname{clos} \operatorname{int} M \tag{2.6}$$

(clos denotes the closure of a set).

If $M \in \mathfrak{M}$ is neither regularly nor singularly traceable, we call it non-traceable.

Finally we define

$$A(M) := \{w \in H^\infty : w(t) \in \operatorname{clos} \operatorname{int} M_t \text{ a.e. on } \mathbb{T}\}.$$

Sometimes the notation $A(M)$ is simply replaced by A.

After these preparations relevant results of [12, 13] concerning the solvability of the considered RHP can be summarized. The sign $\#$ denotes cardinality.

THEOREM 1. *For every $M \in \mathfrak{M}$ the following assertions hold:*

(i) *M is regularly traceable*

$$\begin{aligned} &\Leftrightarrow \exists k \geqslant 0 : \# W_k > 0 \\ &\Leftrightarrow \forall k \geqslant 0 : \# W_k > 0 \\ &\Leftrightarrow \# A > 1 \\ &\Rightarrow \forall k < 0 : \# W_k = 0. \end{aligned}$$

(ii) *M is singularly traceable*

$$\Leftrightarrow \exists k_0 < 0 : \# W_{k_0} > 0$$
$$\Leftrightarrow \# A = 1$$
$$\Leftrightarrow A = W_{k_0} = \{w_0\}, \qquad w_0 \in H^\infty \cap C$$
$$\Rightarrow \forall k \neq k_0 : \# W_k = 0.$$

(iii) *M is nontraceable*

$$\Leftrightarrow \forall k \in \mathbb{Z} : \# W_k = 0$$
$$\Leftrightarrow \# A = 0.$$

The theorem suggests a finer decomposition of $\mathfrak{M}$. For each $k \in \mathbb{Z}_+$ we define

$$\mathfrak{M}_k := \{M \in \mathfrak{M} : \# W_{-k}(M) > 0\}$$

and put

$$\mathfrak{M}_\infty := \{M \in \mathfrak{M} : \# W(M) = 0\}.$$

Thus the class of singularly traceable M is split into several subclasses.

Our first lemma concerns the stability of the relation $M \in \mathfrak{M}_0$ under small perturbations of M.

LEMMA 1. $\mathfrak{M}_0$ is an open subset of $\mathfrak{M}$.

Proof. Let $M_0 \in \mathfrak{M}_0$, $w_0 \in H^\infty \cap C$, $\operatorname{tr} w_0 \subset \operatorname{int} M_0$. Then we have

$$\varepsilon := \inf_{t \in \mathbb{T}} \operatorname{dist}(w_0(t), M_{0t}) > 0.$$

Therefore, for each $M \in U_\varepsilon(M_0)$, it follows that $\operatorname{tr} w_0 \subset \operatorname{int} M$, i.e., $M \in \mathfrak{M}_0$. ▌

Our next intention is to examine how the membership of M to the classes $\mathfrak{M}_k$ is changed by elementary transformations of M. As a shorthand we introduce the notation ($f, g \in C^1(\mathbb{T})$)

$$fM + g := \{(t, w) \in \mathbb{T} \times \mathbb{C} : [f(t)]^{-1} (w - g(t)) \in M_t\}.$$

If $f(t) \neq 0$, then $M \in \mathfrak{M}$ implies that $fM + g \in \mathfrak{M}$. Further

$$\operatorname{tr} w \subset M \Leftrightarrow \operatorname{tr}(fw + g) \subset fM + g$$

and

$$\operatorname{wind}_{fM+g}(fw+g) = \operatorname{wind}_M w + \operatorname{wind} f.$$

LEMMA 2. *Let $1 \leqslant k < \infty$. Then the following implications hold*:

(i) $M \in \mathfrak{M}_0 \Rightarrow tM \in \mathfrak{M}_0$,

(ii) $M \in \mathfrak{M}_k \Rightarrow tM \in \mathfrak{M}_{k-1}$.

Proof. Let $w_0 \in H^\infty \cap C$ be a given function and put $w_1(z) := zw_0(z)$. Then we have

$$\operatorname{tr} w_0 \subset \operatorname{int} M \Rightarrow \operatorname{tr} w_1 \subset \operatorname{int}(tM), \tag{2.7}$$

$$\operatorname{tr} w_0 \subset M, \quad \operatorname{wind}_M w_0 = -k \Rightarrow \operatorname{tr} w_1 \subset tM, \quad \operatorname{wind}_{tM} w_1 = -k+1. \tag{2.8}$$

Assertion (i) follows from (2.7); assertion (ii) can easily be derived from (2.8) and from Theorem 1(ii). ∎

LEMMA 3. *Suppose $M \in \mathfrak{M}_k$ $(1 \leqslant k < \infty)$, $W_{-k}(M) = \{w_0\}$. Then*

(i) $w_0(0) = 0 \Leftrightarrow t^{-1}M \in \mathfrak{M}_{k+1}$,

(ii) $w_0(0) \neq 0 \Leftrightarrow t^{-1}M \in \mathfrak{M}_\infty$.

Proof. (1) If $t^{-1}M \notin \mathfrak{M}_\infty$, careful use of Lemma 2 yields $t^{-1}M \in \mathfrak{M}_{k+1}$. Consequently, either $t^{-1}M \in \mathfrak{M}_{k+1}$ or $t^{-1}M \in \mathfrak{M}_\infty$.

(2) If $w_0(0) = 0$, the function w_1 defined by $w_1(z) := z^{-1}w_0(z)$, $z \in \mathbb{D}$, satisfies $w_1 \in W_{-k-1}(t^{-1}M)$, whence $t^{-1}M \in \mathfrak{M}_{k+1}$.

(3) Let $w_0(0) \neq 0$ and assume $t^{-1}M \in \mathfrak{M}_{k+1}$. Then $W_{-k-1}(t^{-1}M) = \{w_1\}$. The function $\tilde{w}_0$ defined by $\tilde{w}_0(z) := zw_1(z)$ belongs to $W_{-k}(M)$ and satisfies $\tilde{w}_0(0) = 0$. Hence $W_{-k}(M)$ contains at least two elements w_0 and $\tilde{w}_0$. But this is impossible due to Theorem 1(ii). Consequently, we have $t^{-1}M \in \mathfrak{M}_\infty$. ∎

We are now going to investigate problems depending on a real parameter. For this end we consider a family $\{M_\lambda\}_{\lambda \in (0,\infty)}$ of curves which satisfy the following conditions

(i) $0 < \lambda < \infty \Rightarrow M_\lambda \in \mathfrak{M}$,

(ii) $0 < \lambda < \mu < \infty \Rightarrow M_\lambda \subset \operatorname{int} M_\mu$, (2.9)

(iii) The mapping $\mathbb{R}_+ \to \mathfrak{M}$, $\lambda \mapsto M_\lambda$ is continuous.

In [13] existence and dependence on λ of solutions to the family of RHP's $\operatorname{tr} w \subset M_\lambda$ were examined. Since each M_λ belongs to exactly one class $\mathfrak{M}_k$ it is a natural question to ask in which way the index k can change if λ varies.

LEMMA 4. *If* $M_\lambda \in \mathfrak{M}_0$ *then there exists an* $\varepsilon > 0$ *such that* $M_\mu \in \mathfrak{M}_0$ *for all* $\mu \geqslant \lambda - \varepsilon$.

Proof. The assertion immediately follows from Lemma 1 and the above hypotheses (ii) and (iii). ∎

LEMMA 5. *Suppose* $M_\lambda \in \mathfrak{M}_k$ *with* $1 \leqslant k < \infty$. *Then*

(i) $\mu < \lambda \Rightarrow M_\mu \in \mathfrak{M}_\infty$

(ii) $\mu > \lambda \Rightarrow M_\mu \in \mathfrak{M}_0$.

Proof. Since $M_\lambda \in \mathfrak{M}_k$, $0 < k < \infty$, the set $A(M_\lambda)$ contains exactly one element $w_0 \in H^\infty \cap C$. This function satisfies tr $w_0 \subset M_\lambda$(cf. Theorem 1(ii)). The assumption (ii) leads to tr $w_0 \subset \operatorname{int} M_\mu$ if $\mu > \lambda$; for that reason assertion (ii) holds. On the other hand, $A(M_\mu)$ is a subset of $A(M_\lambda)$ for $\mu < \lambda$ and can contain at most the function w_0. But this is impossible because of (2.9), since tr $w_0 \subset M_\lambda$. Thus we have $\# A_\mu = 0$ and assertion (i) follows from Theorem 1(iii). ∎

The next, much deeper, result is, in a sense, a conversion of Lemma 5.

LEMMA 6. *If* $M_{\lambda_1} \in \mathfrak{M}_\infty$ *and* $M_{\lambda_2} \in \mathfrak{M}_0$ *then there exist exactly one* λ_0 *with* $\lambda_1 < \lambda_0 < \lambda_2$ *and exactly one* $k_0 > 0$ *such that the following implications hold*:

$$\lambda < \lambda_0 \Rightarrow M_\lambda \in \mathfrak{M}_\infty,$$
$$\lambda = \lambda_0 \Rightarrow M_\lambda \in \mathfrak{M}_{k_0},$$
$$\lambda > \lambda_0 \Rightarrow M_\lambda \in \mathfrak{M}_0.$$

Proof. We put $\lambda_0 := \inf\{\lambda : M_\lambda \in \mathfrak{M}_0\}$. Then, by Lemma 4, $M_{\lambda_0} \in \mathfrak{M}_k$ $(1 \leqslant k \leqslant \infty)$. Lemma 5 implies $M_\lambda \in \mathfrak{M}_\infty$ if $\lambda < \lambda_0$ while Lemma 4 gives $M_\lambda \in \mathfrak{M}_0$ if $\lambda > \lambda_0$. Now the assertion follows from the proof of Theorem 3 in [13]. ∎

3. GENERALIZED BEST APPROXIMATION BY HOLOMORPHIC FUNCTIONS

In this section the above results will be applied to prove existence and uniqueness of the generalized best approximation of f in H^∞. This means that we seek a $w_0 \in H^\infty$ satisfying

$$d_F(f, w_0) = d_F(f, H^\infty) := \inf_{w \in H^\infty} d_F(f, w)$$

with

$$d_F(f, w) := \operatorname*{ess\,sup}_{t \in \mathbb{T}} F(t, w(t)).$$

We suppose that the distance function F satisfies the following assumptions:

$$F \in C(\mathbb{T} \times \mathbb{C}), \tag{3.1}$$

$$F \in C^1((\mathbb{T} \times \mathbb{C}) \setminus \operatorname{tr} f), \tag{3.2}$$

$$F(t, f(t)) = 0, \qquad \forall t \in \mathbb{T}, \tag{3.3}$$

$$F(t, w) > 0, \qquad \forall (t, w) \in (\mathbb{T} \times \mathbb{C}) \setminus \operatorname{tr} f, \tag{3.4}$$

$$\left| \frac{\partial F}{\partial w}(t, w) \right| > 0, \qquad \forall (t, w) \in (\mathbb{T} \times \mathbb{C}) \setminus \operatorname{tr} f, \tag{3.5}$$

$$\forall C \in \mathbb{R}, \quad \exists C_1 \in \mathbb{R} : t \in \mathbb{T}, \qquad |w| \geqslant C_1 \Rightarrow F(t, w) \geqslant C. \tag{3.6}$$

First of all we fix the function f and denote the class of all F satisfying (3.1)–(3.6) by $\mathfrak{F}$. For $F \in \mathfrak{F}$ let

$$M_\lambda^F := \{(t, w) \in \mathbb{T} \times \mathbb{C} : F(t, w) = \lambda\}.$$

By introducing the system $\{V_\varepsilon(F_0)\}_{\varepsilon > 0}$ of neighborhoods

$$V_\varepsilon(F_0) := \{F \in \mathfrak{F} : M_\lambda^F \in U_\varepsilon(M_\lambda^{F_0}), \forall \lambda \in (0, \infty)\}$$

of $F_0 \in \mathfrak{F}$, the set $\mathfrak{F}$ becomes a Hausdorff space. Note that the mappings

$$\Phi_\lambda : \mathfrak{F} \to \mathfrak{M}, \quad F \mapsto M_\lambda^F \tag{3.7}$$

are continuous.

The classical distance function

$$F(t, w) := |w - f(t)|$$

belongs to $\mathfrak{F}$ if $f \in C^1(\mathbb{T})$.

To avoid trivialities, in the sequel it will be assumed that $f \notin H^\infty$.

LEMMA 7. *If $f \notin H^\infty$ then $d_F(f, H^\infty) > 0$ for each $F \in \mathfrak{F}$.*

Proof. From the assumptions on F one can easily deduce

$$\forall t \in \mathbb{T}, \quad \forall \varepsilon > 0, \quad \exists \delta > 0 : F(t, w) < \delta \Rightarrow |w - f(t)| < \varepsilon. \tag{3.8}$$

The existence of a sequence $\{w_n\} \subset H^\infty$ with $d_F(f, w_n) \to 0$ gives that

$$\forall \delta > 0, \quad \exists n_0 \in \mathbb{Z}_+ : n \geqslant n_0 \Rightarrow \operatorname*{ess\,sup}_{t \in \mathbb{T}} F(t, w_n(t)) < \delta. \tag{3.9}$$

From (3.8), (3.9), and the compactness of $\mathbb{T}$ we get that

$$\forall \varepsilon > 0, \quad \exists n_0 \in \mathbb{Z}_+ : n \geqslant n_0 \Rightarrow \|w_n - f\|_{L^\infty(\mathbb{T})} < \varepsilon,$$

i.e., the convergence of w_n to f in $L^\infty(\mathbb{T})$. But this is only possible if $f \in H^\infty$. ▮

The next theorem shows the existence and the uniqueness of the best approximation. Moreover, it characterizes the nearest function $w_0 \in H^\infty$ to f.

THEOREM 2. *For each $F \in \mathfrak{F}$ the following assertions hold:*

(i) *There exists a unique function $w_0 \in H^\infty$ satisfying*

$$d_F(f, w_0) = d_F(f, H^\infty). \tag{3.10}$$

(ii) *A function $w_0 \in H^\infty$ is the best approximation for f (in the sense of (3.10)) if and only if*

$$w_0 \in H^\infty \cap C, \tag{3.11}$$

$$\operatorname{wind}(w_0 - f) =: -k_0 < 0, \tag{3.12}$$

$$F(t, w_0(t)) = \text{const.} \tag{3.13}$$

Remark 1. Relation (3.11) can be replaced by $w_0 \in H^\infty \cap W_p^1$ $(1 < p < \infty)$, W_p^1 being the Sobolev space on $\mathbb{T}$.

Remark 2. The equations (3.12) and (3.13) coincide with the characterization of the best approximation given by Helton and Howe, because

$$\operatorname{wind}(w_0 - f) = -\operatorname{wind} \frac{\partial F}{\partial w}(\cdot, w_0).$$

Proof. (1) Lemma 7 implies that $M_\lambda^F \in \mathfrak{M}_\infty$ if $\lambda < d := d_F(f, H^\infty)$.

(2) For $\lambda > \sup_{t \in \mathbb{T}} F(t, 0)$ we have $\operatorname{tr} 0 = \mathbb{T} \times \{0\} \subset \operatorname{int} M_\lambda^F$, hence $M_\lambda^F \in \mathfrak{M}_0$.

(3) Lemma 6 in conjunction with the first two steps ensures the existence of λ_0 and $k_0 > 0$ with

$$\begin{aligned} M_\lambda^F &\in \mathfrak{M}_\infty \quad &&\text{if} \quad \lambda < \lambda_0, \\ M_\lambda^F &\in \mathfrak{M}_{k_0} \quad &&\text{if} \quad \lambda = \lambda_0, \\ M_\lambda^F &\in \mathfrak{M}_0 \quad &&\text{if} \quad \lambda > \lambda_0. \end{aligned}$$

From Theorem 1 it follows that $d = \lambda_0$. Obviously, the only function w_0 in

A_{λ_0} is the unique solution of the approximation problem. Theorem 1(ii) gives (3.11)–(3.13).

(4) Conversely, let the function w_0 fulfill (3.11) (3.13). With $\lambda_0 := F(t, w_0(t))$, the function w_0 is a solution of the RHP $\operatorname{tr} w \subset M^F_{\lambda_0}$, and from (3.13) one sees that $w_0 \in W_{-k_0}(M^F_{\lambda_0})$. Therefore we have $M^F_{\lambda_0} \in \mathfrak{M}_{k_0}$ with $k_0 \geqslant 1$. Lemma 5 gives $M^F_\lambda \in \mathfrak{M}_0$ for $\lambda > \lambda_0$ and $M^F_\lambda \in \mathfrak{M}_\infty$ for $\lambda < \lambda_0$. Consequently w_0 is the best approximation of f. ∎

Our final concern is the influence of small perturbations to the best approximation. We first think of f as being fixed and only of the distance function F as being subject to small variations.

For given $F \in \mathfrak{F}$ we denote by w_F the solution of the approximation problem. The set $\mathfrak{F}$ is decomposed into the classes

$$\mathfrak{F}_k := \{F \in \mathfrak{F} : \operatorname{wind}(w_F - f) = -k\}, \qquad k = 1, 2, ...$$

A conjecture raised by Helton [5, p. 362] states that generic F should belong to $\mathfrak{F}_1$. The next theorem (and Theorem 4) confirms this expectation (for functions F in $\mathfrak{F}$).

THEOREM 3. *The set $\mathfrak{F}_1$ is an open dense subset of $\mathfrak{F}$.*

Proof. (1) Suppose $F_0 \in \mathfrak{F}_1$, i.e.,

$$M^{F_0}_{d_0} \in \mathfrak{M}_1, \tag{3.14}$$

where $d_0 := d_{F_0}(f, H^\infty)$. From Lemma 2 we infer that $tM^{F_0}_{d_0} \in \mathfrak{M}_0$. From Lemma 4 one can now conclude the existence of a positive number ε such that

$$tM^{F_0}_{d_0 - \varepsilon} \in \mathfrak{M}_0.$$

The continuity of the maps (3.7) along with Lemma 1 guarantees the existence of $\delta > 0$ such that $tM^F_{d_0 - \varepsilon} \in \mathfrak{M}_0$ for each $F \in V_\delta(F_0)$. Hence,

$$F \in V_\delta(F_0), \qquad \lambda \geqslant d_0 - \varepsilon \Rightarrow tM^F_\lambda \in \mathfrak{M}_0. \tag{3.15}$$

By reducing δ, if necessary, one can achieve that

$$F \in V_\delta(F_0) \Rightarrow M^F_{d_0 - \varepsilon} \subset \operatorname{int} M^{F_0}_{d_0}. \tag{3.16}$$

Combining of (3.14) and (3.16) gives

$$F \in V_\delta(F_0) \Rightarrow M^F_{d_0 - \varepsilon} \in \mathfrak{M}_\infty,$$

and this implies

$$F \in V_\delta(F_0) \Rightarrow \lambda_F := d_F(f, H^\infty) > d_0 - \varepsilon.$$

Therefore (3.15) applies for $\lambda=\lambda_F$. The result is that

$$F\in V_\delta(F_0)\Rightarrow tM^F_{\lambda_F}\in\mathfrak{M}_0. \tag{3.17}$$

On the other hand, we have $M^F_{\lambda_F}\in\mathfrak{M}_k$ with $1\leqslant k=k(F)<\infty$ and therefore Lemma 2 leads to

$$F\in V_\delta(F_0)\Rightarrow tM^F_{\lambda_F}\in\mathfrak{M}_{k-1}. \tag{3.18}$$

Comparing (3.17) and (3.18) we find $k=1$, hence $M^F_{\lambda_F}\in\mathfrak{M}_1$, and thus $V_\delta(F_0)\subset\mathfrak{F}_1$, i.e., $\mathfrak{F}_1$ is an open subset of $\mathfrak{F}$.

(2) It remains to prove that $\mathfrak{F}_1$ is dense in $\mathfrak{F}$. For this end let $F_0\in\mathfrak{F}$ be a function which is not in $\mathfrak{F}_1$. Then $F_0\in\mathfrak{F}_k$, with $2\leqslant k<\infty$. Put $d_0:=d_{F_0}(f, H^\infty)$.

We choose a real-valued function $\eta\colon\mathbb{R}_+\to\mathbb{R}_+\cup\{0\}$ satisfying the relations

$$\eta\in C^\infty,\qquad 0\leqslant\eta(x)\leqslant 1,\qquad \forall x\in\mathbb{R}_+,$$
$$\eta(x)=0,\qquad \forall x\in(0, d_0/2),\qquad \eta(x)=1,\qquad \forall x\in(d_0,\infty).$$

If the positive number ε is sufficiently small then the function F_ε defined by

$$F_\varepsilon(t, w):=F_0(t, w-\varepsilon\eta(F_0(t, w-\varepsilon t^{-1}))\, t^{-1})$$

belongs to $\mathfrak{F}$; moreover $F_\varepsilon\in V_\varepsilon(F_0)$. Note that

$$M^\varepsilon_{d_0}=M^0_{d_0}+\varepsilon t^{-1}, \tag{3.19}$$

with the abbreviation $M^\varepsilon_{d_0}=M^{F_\varepsilon}_{d_0}$, $\varepsilon\geqslant 0$.

From Lemma 2 we obtain

$$M^0_{d_0}\in\mathfrak{M}_k,\ tM^0_{d_0}\in\mathfrak{M}_{k-1},\ \ldots,\ t^kM^0_{d_0}\in\mathfrak{M}_0. \tag{3.20}$$

In particular, the relation

$$tM^0_{d_0}\in\mathfrak{M}_{k-1} \tag{3.21}$$

(with $k-1\geqslant 1$) shows that the RHP tr $w\subset tM^0_{d_0}$ has a unique solution w_0. An application of Lemma 3 to $M:=tM^0_{d_0}$ gives $w_0(0)=0$. Relation (3.19) implies that $w_\varepsilon:=w_0+\varepsilon$ is the only solution of the RHP tr $w\subset tM^\varepsilon_{d_0}$. Since $w_\varepsilon(0)=\varepsilon\neq 0$, we have $M^\varepsilon_{d_0}\in\mathfrak{M}_\infty$ (see Lemma 3 again). This yields

$$d_\varepsilon:=d_{F_\varepsilon}(f, H^\infty)>d_0, \tag{3.22}$$

which together with (3.21) and Lemma 5 implies

$$tM^\varepsilon_{d_\varepsilon}\in\mathfrak{M}_0. \tag{3.23}$$

By definition (3.22) we have $M^{\varepsilon}_{d_\varepsilon} \in \mathfrak{M}_n$ $(1 \leqslant n < \infty)$, and by Lemma 2 it follows that $tM^{\varepsilon}_{d_\varepsilon} \in \mathfrak{M}_{n-1}$. Comparing this with (3.23) one obtains $n = 1$, i.e., $F_\varepsilon \in \mathfrak{F}_1$ for each $\varepsilon > 0$. ∎

In connection with certain applications a slightly modified concept of perturbation is suggested. We fix a distance function F satisfying the conditions (3.1)–(3.6) with respect to $f \equiv 0$ and define the distance $d(f, H^\infty)$ of an arbitrary function $f \in C^1(\mathbb{T})$ to H^∞ by

$$d(f, H^\infty) := \inf_{w \in H^\infty} \operatorname{ess\,sup}_{t \in \mathbb{T}} F(t, w(t) - f(t)).$$

The best approximation of f in this sense is denoted by w_f. We introduce the subclasses $\mathfrak{C}_k$ of $C^1(\mathbb{T})$ by

$$\mathfrak{C}_k := \{f \in C^1(\mathbb{T}) \setminus H^\infty : \operatorname{wind}(w_f - f) = -k\}.$$

In addition, we put $\mathfrak{C}_0 = H^\infty \cap C^1$. This produces the decomposition $C^1(\mathbb{T}) = \mathfrak{C}_0 \cup \mathfrak{C}_1 \cup \mathfrak{C}_2 \cup \cdots$. Again only the class $\mathfrak{C}_1$ is generic:

THEOREM 4. *$\mathfrak{C}_1$ is an open dense subset of $C^1(\mathbb{T})$.*

Proof. (1) As can easily be checked, for $f_0 \in C^1(\mathbb{T}) \setminus \mathfrak{C}_0$ there exists an $\varepsilon > 0$, such that $d(f, H^\infty) \geqslant \delta > 0$ provided that $\|f - f_0\|_{C^1} \leqslant \varepsilon$. Therefore a perturbation of f_0 can be interpreted as variation of the distance function F for fixed f_0. This problem was already considered above.

(2) For $f_0 \in \mathfrak{C}_0$ we define $f_\alpha(t) := f_0(t) + \alpha t^{-1}$ and show that $f_\alpha \in \mathfrak{C}_1$, for each $\alpha \in \mathbb{C} \setminus \{0\}$. Let

$$M^\alpha_\lambda := \{(t, w) \in \mathbb{T} \times \mathbb{C} : F(t, w - f_0(t) - \alpha t^{-1}) = \lambda\}.$$

The functions $w_\alpha := tf_0 + \alpha$ extend holomorphically into $\mathbb{D}$ and satisfy $w_\alpha(t) \in \operatorname{int} tM^\alpha_\lambda$. Therefore

$$tM^\alpha_\lambda \in \mathfrak{M}_0, \qquad \forall \lambda > 0. \tag{3.24}$$

Since f_α (with $\alpha \neq 0$) does not belong to H^∞ we have $\lambda_\alpha := d(f_\alpha, H^\infty) > 0$. By the definition of λ_α, one gets $M^\alpha_{\lambda_\alpha} \in \mathfrak{M}_k$ $(1 \leqslant k < \infty)$ and taking into account (3.24) we conclude from Lemma 2 that $M^\alpha_{\lambda_\alpha} \in \mathfrak{M}_1$. This means $f_\alpha \in \mathfrak{C}_1$ ∎

Remark. In a similar manner the implication

$$k \neq 1, \quad f \in \mathfrak{C}_k, \quad \alpha \in \mathbb{C} \setminus \{0\} \Rightarrow f + \alpha t^{-1} \in \mathfrak{C}_1$$

can be proved.

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